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FEATURES OF FAILURE WAVES IN HIGHLY-HOMOGENEOUS BRITTLE MATERIALS†

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To describe the deformation and evolution of damage of glassy brittle materials, a kinetic model, which takes into account the transformation of elastic energy into surface energy, is proposed. The failure kinetics are characterized by a power dependence on the dynamic overload, which is equal to the difference between the rates of change of elastic and surface energies relative to the increase in damage of the medium. The model is applied to the problem of a plane failure wave in a half-space arising from the application of a normal load to the boundary. An approximate asymptotic solution is constructed by combining the two power series for the regions of slow and rapid change of the solution. As found in previous experiments, at moderate loads the values of the velocity and longitudinal stress in the regions of elasticity and the failed state of the material are the same. As the load increases, the distribution of these quantities become two-wave, the amplitude of the forerunner being greater than the elastic limit under uniaxial compression. In that case the structure of the failure wave largely depends on the power index of the kinetic function in the neighbourhood of the static state. If the index is less than one, the kinetics exerts an influence only in a finite neighbourhood of the failure front. © 1998 Elsevier Science Ltd. All rights reserved.

Experimental investigation [1–3] of the processes of dynamic failure of certain glasses has revealed various characteristic features of the plane compression waves. The most important of these include single-wave longitudinal stresses, two-wave transverse stresses, the initiation of failure waves at points on the boundary of the body or contact surfaces and the absence of damage at interior points of the body at high stress levels until the approach of the failure wave.

These features cannot be modelled using the conventional description of a failure wave [4–6],‡ which inevitably gives a two-front shock wave structure with an amplitude of the forerunner equal to the elastic limit of the material.

When investigating the features of the deformation and failure of highly homogeneous brittle materials here, we have taken into account that the absence of damage at interior points of the body, even at a high level of stresses, before the failure wave approaches appears to be due to the initial lack of sufficiently large microdefects and the time that they take to form, which is longer than the transit time of acoustic waves through the body. Unlike the internal points of a body, its boundary, even a well-machined boundary, contains a large number of macroscopic cracks and acts as the geometric site of starting points for failure waves. Thus, the problem of the dynamic failure of highly-homogeneous brittle bodies reduces to investigating the distribution of a fragmentation wave through a pre-stressed elastic medium in which the stress level is governed both by the load on the boundary and by the internal properties of the fragmented material.

Another important feature that we take into account is the need to interpret the experimental results on models with a finite failure kinetics. The problem with an instantaneous kinetics in which the damage changes simultaneously with the stresses is that typically the solution for wave propagation in an inelastic half-space under a dynamic load on the boundary is known to be non-unique. The conventional method of dealing with this problem in cases where the dependence of the stresses on the strains during loading is represented by a single curve with both elastic and inelastic parts is to require that the corrected solution must be continuous in relation to a small variation of the curve, and in particular to smoothing of the corner point corresponding to the transition from the elastic to the inelastic state. The situation here is different: after the approach of the fragmentation wave, the material changes from an elastic state in which the stress level bears no relation to the elastic limit to a state with a different relation between the stresses and strains. Non-uniqueness of the solution is not eliminated by varying these separate, unassociated curves. This is why kinetics must be used to regularize the solution.

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‡See also Nikolayevskii, V. N., Maximum velocity of the failure front and dynamic overload of brittle materials. Preprint No. 123. Inst. Problem Mekh. Akad. Nauk SSSR, Moscow, 1979.

1. BASIC EQUATIONS

In both its initial and its failed state, the material is assumed to be macroscopically homogeneous and initially isotropic, and thermal effects are assumed to be small. The strains, measured from the natural (unloaded and undamaged) state, are characterized by the symmetric tensor of small strains $\mathbf{e}$, and the degree of damage (failure) of the material is characterized by the scalar parameter ω. The quantities $(\mathbf{e}, \omega)$ characterize the state, the reaction of the material is given by the functions

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{e}, \omega), \quad U = U(\mathbf{e}, \omega)$$

where $\boldsymbol{\sigma}$ is the symmetric stress tensor, U is the elastic potential of the damaged medium, defined by the expression [7, 8]

$$\rho U(\mathbf{e}, \omega) = \tfrac{1}{2} K I_1 + \mu J - \alpha_p I_1 \omega - \alpha_s J \omega + \gamma\omega + \tfrac{1}{2}\beta\omega^2 \quad (1.1)$$

$$I_1 = \mathbf{e}:\mathbf{I}, \quad J = (\mathbf{e}':\mathbf{e}')^{1/2}, \quad \mathbf{e}' = \mathbf{e} - \tfrac{1}{3} I_1 \mathbf{I}$$

where ρ is the density of the material, K and μ are the bulk compression and shear moduli, the parameters α_p, $\alpha_s > 0$ characterize the decrease of accumulated elastic energy due to failure, γ, $\beta > 0$ are characteristics of the effective surface energy of a unit mass of the failed material and $\mathbf{e}'$ is the stress tensor deviator.

The stress tensor associated with the elastic potential $\boldsymbol{\sigma} = \rho\partial U/\partial \mathbf{e}$ is given by the expression

$$\boldsymbol{\sigma} = (K I_1 - \alpha_p \omega)\mathbf{I} + (2\mu - \alpha_s \omega / J)\mathbf{e}' \quad (1.2)$$

The evolution of damage of the material is defined by the kinetic equation

$$\dot{\omega} = \frac{1}{\tau}\Phi\left\{-\frac{1}{\beta}\frac{\partial U}{\partial \omega}\right\}, \quad \Phi(z) = \begin{cases} A z^k, & z > 0 \\ 0, & z \leqslant 0 \end{cases}, \quad k > 0 \quad (1.3)$$

where $\tau > 0$ is the characteristic time.

Using expression (1.1), we can write the kinetic equation (1.3) in the form

$$\dot{\omega} = \tau^{-1}\Phi\{\beta^{-1}(\alpha_p I_1 + \alpha_s J - \gamma - \beta\omega)\} \quad (1.4)$$

The choice of (1.4) as the law of change of the damageability can be justified as follows.

In a slow process ($\omega \to 0$), relation (1.4) yields the model of a damaged body with instantaneous kinetics [7, 8] based on local balance of the accumulated elastic and effective surface energies.

The quantity $-\partial U/\partial\omega$, which is equal to the difference of the rates of change of the elastic and surface energies relative to the increase in damage, is a natural scalar measure of the "dynamic overload", applicable to different types of stress–strain state.

The main qualitative features of diffuse failure, including, in particular, threshold values of the strains at which damage starts to accumulate, failure during both tension and shear, and the effects of dilatancy and internal friction, can be described by the single law (1.4).

It is clear from Eq. (1.4) that if the kinetics of failure is taken into account, we obtain the model of a material with a long decaying memory of previous states, since with the given history of deformation the solution of (1.4) is a functional defined in $\mathbf{e}(\xi)$, $\xi \leqslant t$ and which depends parametrically on t.

2. FAILURE UNDER UNIAXIAL COMPRESSION

We will apply the model to the uniaxial deformation of an initially unperturbed half-space $x_1 \geqslant 0$ which is damaged by the operation on the boundary $x_1 = 0$ of a normal stress $\sigma_{11} = -p_0 H(t)$, where $p_0 = \text{const} > 0$, $H(t)$ is the Heaviside function.

The system of equations in the unknowns $(\upsilon_1, \sigma_{11}, \omega)$, where υ_1 is the velocity of a particle along the x_1 axis, can be written, using (1.2) and (1.4) in the form

$$\frac{\rho\partial\upsilon_1}{\partial t} - \frac{\partial\sigma_{11}}{\partial x_1} = 0, \quad \frac{\partial\sigma_{11}}{\partial t} - \Lambda_0\frac{\partial\upsilon_1}{\partial x_1} = -\alpha\tau^{-1}\Phi(z), \quad \frac{\partial\omega}{\partial t} = \tau^{-1}\Phi(z) \quad (2.1)$$

$$\Lambda_0 = K + \frac{4\mu}{3}, \quad \alpha = \alpha_p - \alpha_s\sqrt{\frac{2}{3}} < 0, \quad \Lambda_f = \Lambda_0 - \frac{\alpha^2}{\beta}, \quad z = \frac{\alpha\sigma_{11}}{\beta\Lambda_0} - \frac{\Lambda_f\omega}{\Lambda_0} - \frac{\gamma}{\beta}$$

with zero initial data

$$v_1(x_1,0)=\sigma_{11}(x_1,0)=\omega(x_1,0)=0,\quad x_1\geqslant 0 \tag{2.2}$$

and the boundary condition

$$\sigma_{11}(0,t)=-p_0,\quad t\geqslant 0 \tag{2.3}$$

It will be useful to introduce dimensionless variables

$$\bar{t}=\frac{t}{t_0},\quad \bar{x}=\frac{x_1}{c_0t_0},\quad v=\frac{v_1}{c_0},\quad \bar{\sigma}=\frac{\sigma_{11}}{\Lambda_0},\quad \bar{\alpha}=\frac{\alpha}{\Lambda_0},$$

$$\bar{b}^2=\frac{\Lambda_f}{\Lambda_0},\quad \bar{a}=\frac{\alpha}{\beta},\quad \bar{g}=\frac{\gamma}{\beta},\quad \bar{\tau}=\frac{\tau}{t_0},$$

where t_0 is the characteristic time, $c_0 = (\Lambda_0/\rho)^{1/2}$ is the velocity of longitudinal elastic waves, and the quantities $\bar{\alpha}$ and $\bar{a}$ are related by the identity $\overline{\alpha a} + \bar{b}^2 = 1$.

In these dimensionless variables, system (2.1) can be written in the form

$$\dot{v}-\partial\sigma/\partial x=0,\quad \tau(\dot{\sigma}-\partial v/\partial x)=-\alpha\Phi(z),\quad \tau\dot{\omega}=\Phi(z) \tag{2.4}$$

$$z=a\sigma-b^2\omega-g$$

Here and below the bar above dimensionless variables is omitted.

System (2.4) is defined in the interval $0\leqslant x\leqslant x_*(t)$, where the unknown function $x_*(t)$ is found from the solution and gives the position of the failure wave front moving away from the boundary $x = 0$. For $x_*(t) < x < t$, the material is in an elastic intact state ($\omega = 0$) and its behaviour is described by the first two equations of system (2.4) with zero right-hand side

$$\partial v^0/\partial t-\partial\sigma^0/\partial x=0,\quad \partial\sigma^0/\partial t-\partial v^0/\partial x=0. \tag{2.5}$$

The boundary and initial conditions (2.2) and (2.3) are written in the form

$$\sigma(0,t)=-p_0,\quad t\geqslant 0;\quad \sigma^0(x,0)=v^0(x,0)=0,\quad x\geqslant 0 \tag{2.6}$$

To these we add the conditions for the solution to be matched on the fracture wave front

$$\sigma(x,t)=\sigma^0(x,t),\quad v(x,t)=v^0(x,t),\quad x=x_*(t) \tag{2.7}$$

For $t \gg \tau$ we will consider the approximate solution of boundary-value problem (2.4)–(2.7) in the form of power series. Since (2.4) is a hyperbolic system with a small parameter in the highest derivatives and when $\tau = 0$ it is also a second-order hyperbolic system with the same number of boundary conditions but, unlike (2.4), allows solutions with discontinuities which differ from elastic shock waves, we shall construct a solution which is a combination of two series, as in [9]. The first, a power series in the small parameter $\delta(\tau)$, will be used in regions in which the solution has low gradients and the second expansion in the small parameter $\Delta(\tau)$ will be used in the large gradient zone corresponding to the neighbourhood of a shock wave.

Representing the solution vector $w = (v, \sigma, \omega)$ in the form of the power series

$$w=w_0+w_1\delta+w_2\delta^2+\ldots,\quad \delta=\tau^{1/k}$$

where k is the exponent of the kinetic function (1.3), we find

$$z=z_0+z_1\delta+z_2\delta^2+\ldots,\quad z_0=a\sigma_0-b^2\omega_0-g,\quad z_i=a\sigma_i-b^2\omega_i,\quad i\geqslant 1$$

It follows from (2.4) that $\Phi(z_0) = 0$, that is $z_0 = 0$. This means that, in the zero approximation, the solution is determined by equations of a medium with instantaneous failure kinetics

$$\dot{v}_0-\partial\sigma_0/\partial x=0,\quad \dot{\sigma}_0-b^2\partial v_0/\partial x=0 \tag{2.8}$$

In this approximation, the velocity of the failure wave front is equal to $b = (\Lambda_f/\Delta_0)^{1/2}$, and its position is given by the equation $x = bt$.

Systems (2.5) and (2.8) with edge and boundary conditions (2.6), (2.7) have a one-parameter family of solutions with a strong discontinuity on the line $x = bt$.

$$\sigma^0(x,t) = -v^0(x,t) = \sigma_*, \quad bt \leqslant x \leqslant t \tag{2.9}$$
$$\sigma(x,t) = -p_0, \quad v(x,t) = ((1-b)\sigma_* + p_0)/b, \quad 0 \leqslant x \leqslant bt$$

where σ_* is the arbitrary amplitude of the elastic forerunner.

In order to construct a rapidly-changing solution in the neighbourhood of the front $x = bt$, we introduce new independent variables to "extend" the solution in a direction perpendicular to the front

$$\xi = t, \quad \eta = (x - bt)/\Delta$$

where Δ is a small parameter for which $\tau = \Delta^{k+1}$. Since

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} - \frac{b}{\Delta}\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial x} = \frac{1}{\Delta}\frac{\partial}{\partial \eta}$$

using the solution vector (u, z), where u is the displacement of the particle for which

$$v = \frac{\partial u}{\partial t}, \quad \sigma = b^2\frac{\partial u}{\partial x} + \alpha z + \alpha g, \quad \omega = a\frac{\partial u}{\partial x} - z - g$$

system (2.4) can be written in the form of the two equations

$$u_{\xi\xi} - 2\frac{b}{\Delta}u_{\xi\eta} - \frac{\alpha}{\Delta}z_\eta = 0; \quad \frac{\tau a}{\Delta}u_{\xi\eta} - \frac{\tau a b}{\Delta^2}u_{\eta\eta} - \tau z_\xi + \frac{\tau b}{\Delta}z_\eta = \Phi(z)$$

Using the expansions

$$u = u^{(0)}\Delta + u^{(1)}\Delta^2 + \ldots, \quad z = z^{(0)}\Delta + z^{(1)}\Delta^2 + \ldots$$

and equating terms of the same order of smallness, we obtain the above relation for $\Delta(\tau)$ and a system of equations for the zero approximation which can be reduced to a non-linear parabolic equation in $e = u_\eta^{(0)}$

$$\frac{\partial e}{\partial \xi} = D\frac{\partial}{\partial \eta}\left(\frac{\partial e}{\partial \eta}\right)^{1/k}, \quad D = \frac{|\alpha|}{2b}\left(\frac{|\alpha| b}{A}\right)^{1/k} > 0. \tag{2.10}$$

As $\Delta \to 0$ the domain of definition of the solution (2.10) becomes $\{\xi \geqslant 0, -\infty < \eta \leqslant 0\}$. The initial condition for (2.10) is

$$e(0,\eta) = e_0, \quad -\infty < \eta \leqslant 0, \quad e_0 = -(p_0 + \alpha g)/b^2 \tag{2.11}$$

which follows from relations (2.9) and the formula $\sigma^{(0)} = b^2 u_\eta^{(0)} + \alpha g$.

On the right-hand boundary $\eta = 0$, the boundary condition can be written in the form

$$e(\xi,0) = e_*, \quad \xi \geqslant 0, \quad e_* = \alpha g/\{b(1-b)\} \tag{2.12}$$

following from relations (2.9), the continuity of σ and v on the line $x = bt$ and the condition $\sigma^{(0)} + \upsilon^0 = 0$ in the region of the elastic state of the material.

It is worth noting that the condition of continuity of stress and velocity on the front $x = bt$ uniquely defines the amplitude of the elastic forerunner $\sigma_* = be_* = \alpha g/(1-b)$ and the jumps of the strain and damage on the failure wave front $[e] = \alpha g/b$, $[\omega] = g/b$.

The boundary condition as $\eta \to -\infty$ at the left-hand boundary, which expresses matching with the slowly changing solution (2.9), has the form

$$e(\xi,\eta) = e_0, \quad \xi \geqslant 0, \quad \eta \to -\infty \tag{2.13}$$

Problem (2.10)–(2.13) is self-similar, since D = const, and boundary conditions (2.11)–(2.13) are constant.

Making the substitution $\zeta = \eta/(n^2/D\xi)^n$, $n = k/(k+1)$, we obtain the boundary-value problem

$$e''(\zeta) + (k+1)\zeta(e'(\zeta))^{(2k-1)/k} = 0, \quad -\infty < \zeta \leq 0 \tag{2.14}$$

$$e(-\infty) = be_0, \quad e(0) = e_*$$

When $k = 1$ Eq. (2.14) is identical with the linear heat conduction equation, the solution of which is

$$e(\zeta) = e_0 + \frac{\chi(\zeta)}{b^2}, \quad \chi(\zeta) = \frac{2}{\sqrt{\pi}}(p_0 + \sigma_*)\int_{-\infty}^{\zeta} e^{-x^2}dx \tag{2.15}$$

In the zero approximation the velocity and stress are given by the expression

$$v^{(0)}(\zeta) = -be_0 - \frac{\chi(\zeta)}{b}, \quad \sigma^{(0)}(\zeta) = -p_0 + \chi(\zeta) \tag{2.16}$$

If $k \neq 1$ the general solution (2.14) has the form

$$e(\zeta) = C_2 + \int_0^{\zeta} (C_1 + qx^2)^m dx, \quad q = \frac{k^2-1}{2k}, \quad m = \frac{k}{1-k}$$

When $k < 1$, the quantity $q < 0$ and it follows from the necessary condition for a solution to exist $e' = (C_1 + q\zeta^2)^m > 0$, that the change in the solution between $e(0)$ and $e(-\infty)$ must occur in the finite interval $[\zeta_0, 0]$, $\zeta_0 < 0$. Provided that the matching is smooth $(de/d\zeta|_{\zeta_0} = 0)$, a solution which satisfies boundary conditions (2.12) and (2.13) can be written in the form

$$e(\zeta) = \frac{\sigma_*}{b} - (p_0 + \sigma_*)\frac{I(\zeta/\zeta_0)}{b^2 I(1)} \tag{2.17}$$

$$I(z) = \int_0^z (1-z^2)^m dz, \quad \zeta_0 = -\left\{\frac{p_0 + \sigma_*}{|q|^m b^2 I(1)}\right\}^{1/(2m+1)}, \quad m = \frac{k}{1-k} > 0$$

In that case, the velocity and stress are given by the relations

$$v^{(0)}(\zeta) = -\sigma_* + (p_0 + \sigma_*)\frac{I(\zeta/\zeta_0)}{bI(1)}, \quad \sigma^{(0)}(\zeta) = -\sigma_* - (p_0 + \sigma_*)\frac{I(\zeta/\zeta_0)}{I(1)} \tag{2.18}$$

In the case when $k > 1$, the quantities $q > 0$, $m = k/(1-k) < -1$ and it follows from the condition for a solution to exist $e'(\zeta) > 0$ that $C_1 > 0$, since otherwise the integral in the general solution of Eq. (2.14) will diverge as $\zeta^2 \to -C_1/q$. Since the general solution is bounded for $C_1 > 0$, the change in the solution occurs in the interval $-\infty < \zeta \leq 0$. Taking into account the boundary conditions (2.12) and (2.13), we obtain

$$e(\zeta) = \frac{\sigma_*}{b} - (p_0 + \sigma_*)\frac{J(-\zeta/\zeta_*)}{b^2 J(\infty)} \tag{2.19}$$

$$J(z) = \int_0^z (1+z^2)^m dz, \quad \zeta_* = \left\{\frac{1}{q^m b^2 J(\infty)}(p_0 + \sigma_*)\right\}^{1/(2m+1)}$$

The relations $v^{(0)}(\zeta)$ and $\sigma^{(0)}(\zeta)$ are given in this case by formulae (2.18) in which $I(z)$ must be replaced by $J(z)$.

3. DISCUSSION

The following typical features of the behaviour of brittle highly-homogeneous materials during uniaxial dynamic compression are described by the proposed failure model. Under loading $p_0 < |\sigma_f|$, where $\sigma_f = \Lambda_0\gamma/\alpha < 0$ is the threshold value of the compressive stress, the material remains in an intact elastic state throughout. The pressure p_0 on the boundary lies in the range $\sigma_* - p_0 < \sigma_f$, $\sigma_* = (1 + b)\sigma_f$, the approximate asymptotic solutions for the longitudinal stress and velocity are in the form of a rectangular step

$$\sigma_{11}(x_1,t) = -p_0, \quad v_1(x_1,t) = p_0/\rho c_0, \quad t > 0, \quad 0 < x_1 < c_0 t$$

with identical amplitude in the region of elasticity $bc_0t \leq x_1 \leq c_0t$ and the region $0 < x_1 < bc_0t$ of the failed state of the material. Unlike $\sigma_{11}(x_1, t)$ and $\upsilon_1(x_1, t)$, the profiles of transverse stress σ_{22}, longitudinal strain $e_{11}(x_1, t)$ and damage $\omega(x_1, t)$ have two-wave configurations, such that

$$e_{11} = -p_0/\Lambda_0, \quad \sigma_{22} = \lambda e_{11}, \quad \omega = 0, \quad bc_0t < x_1 \leq c_0t$$

$$e_{11} = -(p_0 + \alpha\gamma/\beta)/\Lambda_f, \quad \sigma_{22} = (\lambda - \alpha\hat{\alpha}/\beta)e_{11} + \hat{\alpha}\gamma/\beta$$

$$\omega = \alpha(e_{11} - \gamma/\alpha)/\beta, \quad 0 \leq x_1 \leq bc_0t$$

where $\hat{\alpha} = \alpha_p + \alpha_s/\sqrt{6}$, and the remaining notation was introduced above.

The single-wave profile $\sigma_{11}(x_1, t)$ is due to the fact that any solution of the form

$$\sigma_{11}(x_1,t) = \begin{cases} \sigma_* = \text{const}, & bc_0t \leq x_1 \leq c_0t \\ -p_0, & 0 \leq x_1 \leq bc_0t \end{cases}$$

with an amplitude of the elastic forerunner $\sigma_* \neq -p_0$ that is acceptable in the approximation of instantaneous kinetics, is asymptotically unstable in a finite failure kinetics. In fact, if the slowly-changing solution has a discontinuity, it is necessary to use a solution with "extended" space variable $\eta = (\chi - bc_0t)/\Delta$ in the neighbourhood of that discontinuity. The existence of such a solution requires a definite elastic forerunner amplitude $\sigma_* = (1 + b)\sigma_f$ and the inequality $e' \geq 0$. Since these conditions are mutually exclusive, for $|\sigma_f| < p_0 < (1 + b)|\sigma_f|$ the only possible solution has a forerunner $\sigma_* \neq -p_0$, and so the profile σ_{11} has a *single-wave* configuration.

For loads applied to the boundary exceeding $(1 + b)\sigma_f$, the asymptotic profile $\sigma_{11}(x_1, t)$ and $\upsilon_1(x_1, t)$ have a two-wave configuration. In that case, the forerunner amplitude is unchanged and equal to σ_* for any pressure p_0 on the boundary. Whatever the form of the boundary conditions at $x_1 = 0$, the structure of the "smoothed" front is determined by the exponent k of the expansion of the kinetic function $\Phi(z)$ in the neighbourhood of zero, where $z = \sigma_{11}/(\beta\Lambda_0) - b^2\omega - \gamma/\beta$ is the dynamic overload of the material. For $k \geq 1$ the fracture kinetics have an effect throughout the region occupied by the damaged material, whereas for $k < 1$ the kinetics is only important in a finite neighbourhood of the failure front. This is illustrated by Fig. 1, which shows the dependence of the stress σ_{11} on the coordinate η for $b = 0.3$ for different k.

As in the case of small loads, the quantities e_{11}, σ_{22}, ω have a discontinuity on the failure wave front, changing smoothly after the front to values corresponding to the equilibrium state. This is illustrated in Fig. 2 by the load trajectories in the plane $\sigma_{11} - e_{11}$ depicted by curves $1(p_0 < (1 + b)|\sigma_f|$ and $2(p_0 > (1 + b)|\sigma_f|$.

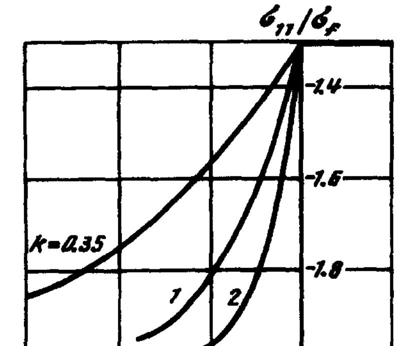

Fig. 1.

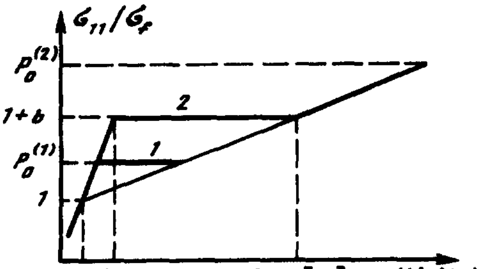

Fig. 2.

The velocity of particles on the failure wave front is independent of the stresses on the boundary and is equal to $(1 + b)|\sigma_f|/(\rho c_0)$, depending only on the quasi-static properties of the material. It is clear from formulae (2.15) and (2.18) that the lines of constant values of the velocity and stresses in the neighbourhood of the failure front are defined by the equation $x - bc_0t = Mt^{k/(1+k)}$, M = const. Thus, instead of being straight lines parallel to the front, they deviate from it, because the large gradient zone expands over time. Defining the effective width of the failure wave as

$$\Delta x = \tfrac{1}{2}(p_0 + (1+b)\sigma_f)\max_{x_1}|\partial\sigma_{11}/\partial x_1|,$$

we see that when $k = 1$ the wave width behaves as $t^{1/2}$ and is independent of the intensity of the applied load. For $k \neq 1$ the increase in width $t^{k/(k+1)}$ with a coefficient which depends on the acting boundary pressure as well as the parameters of the material.

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